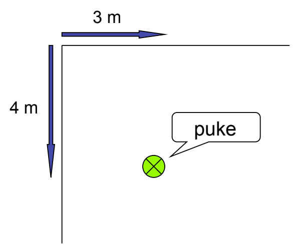
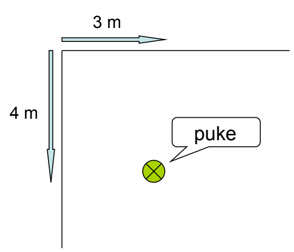
MINKOWSKIAN SPACE, GRAPHS, EVENTS, WORLD LINES and MEASURING TIME

One second after we started our stop watch, little Johnny pukes in the corner of his room. Relative to the corner, the happening occurs 5 meters out. To get to the puddle from the corner, Johnny's mommy goes 3 meters across the north wall, then 4 meters out parallel to the west wall. In other words, in Euclidian space the happening occurred at the Cartesian coordinates (3 meters, 4 meters).



Euclidian geometry is something you are familiar with--it is something you use in your math and physics classes all of the time. When you try to do a similar operation in a relativistic setting, you run into two problems you wouldn't have to deal with in our puking example.



To begin with, Einstein's theory is not set in Euclidian space. The kind of space that does the trick for Einstein is *Minkowskian space* (an explanation of the consequences of this is coming).

Second, in Einstein's world, time is literally a part of the geometry of space. (That is from whence the terms "space-time" or "four-space" came.)

Two problems arise from this observation:

First, if temporal measurements (i.e. time) and spatial measurements (distances) are to be on the same footing, it follows the two must have the *same units*. So how are we going to do that?

The answer is found in the observation that light travels a distance $d=ct$ in time t. How does this help? Let's say a subatomic particle decays after $10^{-8}\ seconds$. During that time, light will travel a distance:

$$d = ct = (3x10^{8}\ m/s)(10^{-8}\ seconds) = 3\ meters.$$

If you were one of those special people *in the know*, and if I wanted to be cryptic instead of blurting out the time in seconds, I could say it took *3 meters of light time*. Again, if you understood the code, you'd realize I was telling you that *the time it took* the particle *to decay* was the same as *the time it takes light to travel 3 meters*.

This unit of *time* is defined as *meters of light time*. As the symbol t is normally associated with time in *seconds*, time in *meters of light time* is depicted as ct, where c is the speed of light in *meters/second* and t is time in *seconds*. What's more, if we dump the "*of light time*" part, we end up with time in units of "*meters*"-- the same as that of spatial measurements. Clever, eh?

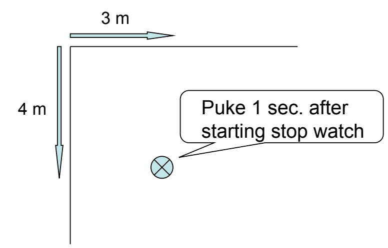
The alternative way to generate units for temporal and spatial measurements that match up is to define time in seconds and define *distances* in what are called *light seconds*. A *light second* is the distance that light travels in one second. Mathematically:

$$d = ct$$

$$\Rightarrow\ (3x10^{8}\ m/s)(1\ second) = 3x10^{8}\ meters.$$

Just as 1 meter of light-time corresponds to a time of $3.3x10^{-9}\ seconds$, 1 light-second corresponds to a distance of $3x10^{8}\ meters$.

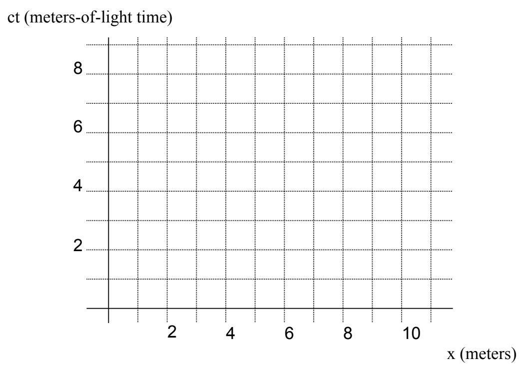
The second problem has to do with the way we describe the position of something in space-time.



It is no longer sufficient to tell someone merely *where* and object is in a spatial sense. We now have to include where it is temporally. As such, a point is no longer a simple point. It is now an *event* (and I'm not being euphemistic here--points on a space-time diagrams are literally called *events*).

In other words, in Einstein's theory, the *event* associated with little Johnny puking (Johnny's mommy might call this a *catastrophic event*, but to Einstein it is merely an *event*) would have the coordinates (3 meters, 4 meters, 300,000,000 meters), where the "300,000,000" would identify the fact that he did the deed *300,000,000 meters of light time* (this is the equivalent of one second) after the stop watch was started.

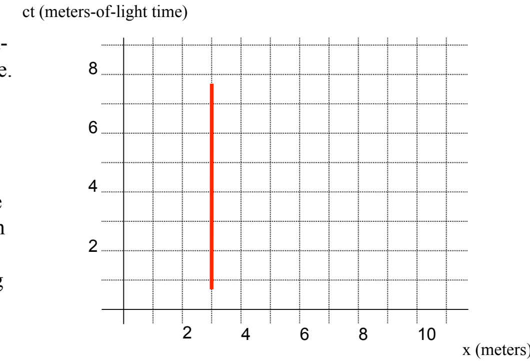
New topic: Trying to deal with *three spatial coordinates* and *one time coordinate* is cumbersome, especial given the fact that if you are normal, thinking in four dimensions is not something you do. So to make life easier, we will be examining situations in which the motion of an object is along one spatial dimension.



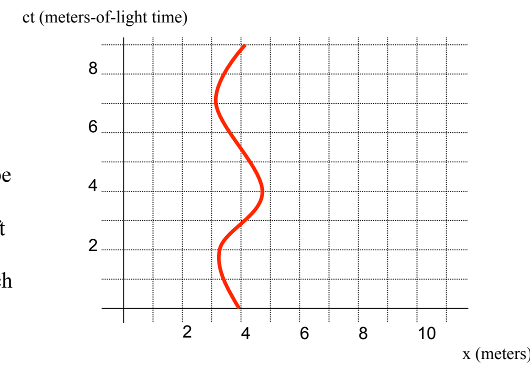
To start with, the plot of an object's motion in any relativistic setting, whether it be two dimensions (one spatial coordinate, one temporal coordinate) or four dimensional (three spatial coordinates, one temporal coordinate) is called the body's *world-line*.

So let's consider the world-line of an ant on a taut wire. What observations can we make?

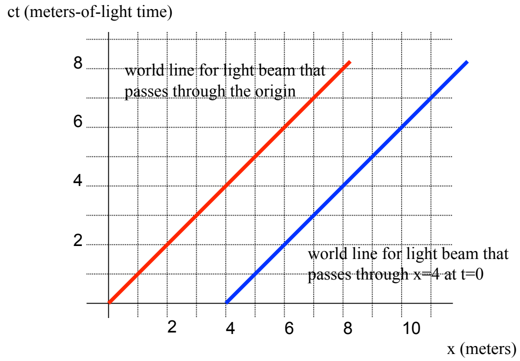
To begin with, the ant's world-line will NEVER be stationary. Yes, the ant can sit still in its wire, but all the while it will be moving in *time*. So if the ant rests at a spatial point three meters from the origin, its world-line would look like the one shown to the right.



The world line of an ant that started at x=4 when the stop-watch started, went streaking left on the wire (this would correspond to a negative slope as the velocity would be negative), then right, then left again, then back right would look something like the sketch shown to the right. (Yes, he sat still for just a moment right around ct =*1.8*, *4.0* and the *7.1 meter of light time*--can you see it?)

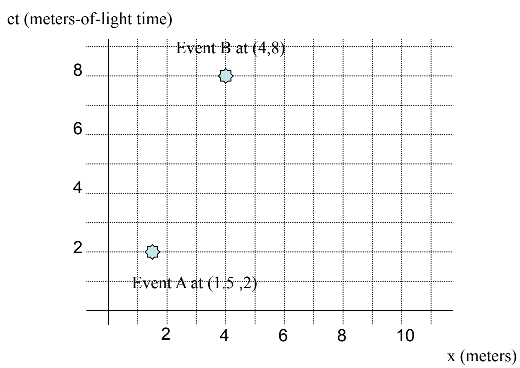


A second point of interest about world lines. What would the world-line for a photon of light look like? Well, the photon would move *3 meters of light time* in the time it took to move *3 meters*, and *7 meters of light time* in the time required to go *7 meters*, so its plot would look like the graph shown.



The important point? As no object will ever travel faster than light, no world line will ever be found in the region below 45^{o} on the plot. That doesn't mean there can't be an event in that lower region. It just means the line defining the motion of a single object in space-time can never have a slope less than 45^{o}.

So let's look at the motion of another ant walking on a taut wire. The claim is that there are two events that are noteworthy. First, after 2 *meters of light time* there is a sneeze. And second, after 8 *meters of light time*, there is another sneeze. Both events are shown and labeled on the sketch.

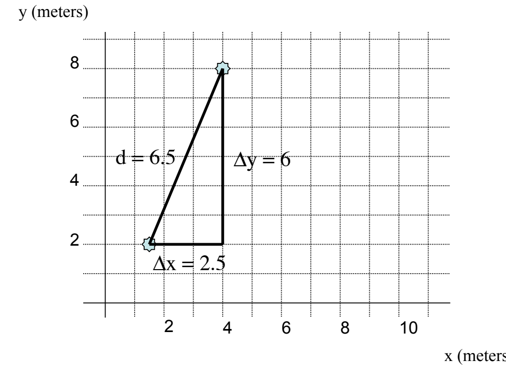


Two points:

FIRST: Could the same ant have generate the two events? (Answer: YES. The angle of the line between the two points is less than 45^{o}.)

SECOND: What is the *interval* between the two events? (If this was Euclidian space, this would be the same as asking what the *distance* was between the two happenings.)

In fact, IF we were looking at the events as plotted in Euclidian space, and if the coordinates were "x" and "y" instead of "x" and "ct," the solution to the "what is the distance between the two points" could easily be answered by drawing the triangle shown on the sketch and using the Pythagorean theorem. With that, we could write:



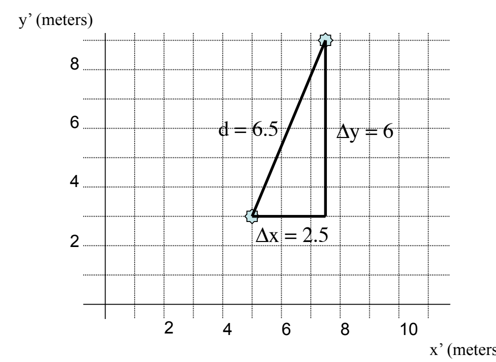
$$\Delta x^2 + \Delta y^2 = d^2$$

$$\Rightarrow \quad d = \left(\Delta x^2 + \Delta y^2\right)^{1/2}$$

$$\Rightarrow \quad d = \left((2.5)^2 + (6)^2\right)^{1/2}$$

$$\Rightarrow \quad d = 6.5 \text{ m}$$

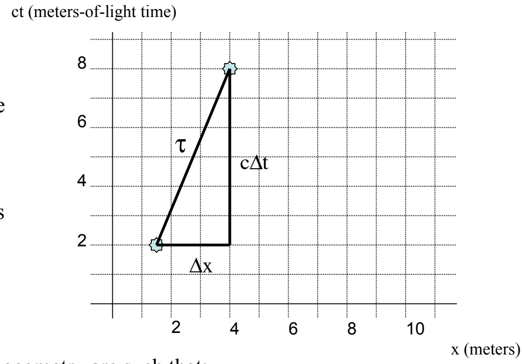
What's more, if we used a second coordinate system that was shifted to the left and down a bit, our events would have different, primed coordinates but the distance between the two happening would still be governed by the Pythagorean relationship, and the distance between the two happenings would still be as calculated by:



$$\Delta x^2 + \Delta y^2 = d^2$$

In both cases, the distance "d" between the happenings would be 6.5. Apparently, distances in Euclidean geometry are not *frame of reference* dependent.

The unfortunate problems we face with Einstein's physics is that the axes are NOT *x* and *y* and the geometry is not Euclidian. In fact, the axes are *x* and *ct* and the geometry is Minkowskian. Minkowskian geometry has some peculiar characteristics, one of which is that the relationship between the sides of a right triangle in that geometry don't obey the Pythagorean relationship. In fact, the relationship between sides of a right triangle in that geometry are such that:

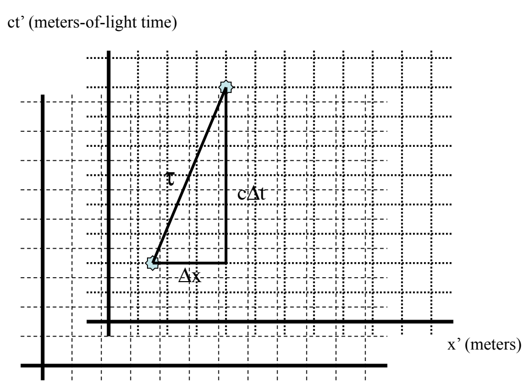


$$\tau^2 = (c\Delta t)^2 - \Delta x^2$$

Note: Because the interval must always be POSITIVE, the terms on the right-side would be reversed if the change of position was greater than the change of time.

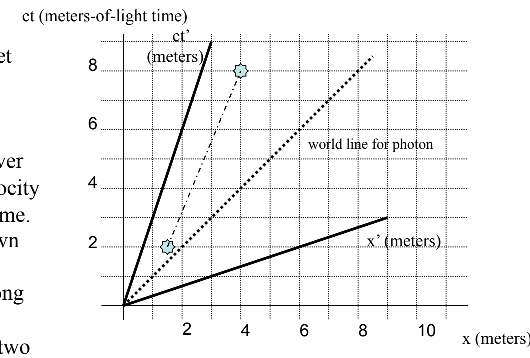
The term "τ" is called the *interval* between the two events. (In fact, this is the mathematical *definition* of the *interval*.)

What is important about intervals is that they don't change when viewed from different, primed reference frames. That is, it doesn't matter what reference frame you choose to take coordinates from, the *interval* will always be calculated to be the same. As such, intervals are said to be *invariant*.



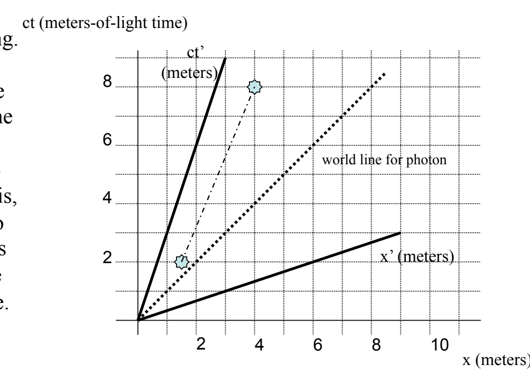
The second noteworthy point about intervals is that they are always *positive*.

So this is where it begins to get exciting. We have the same interval and same unprimed coordinate axes, but let's now assume there is another observer who is moving with some velocity *v,* relative to our unprimed frame. That observer will have his own set of clocks and meter sticks which will also be moving along with velocity *v*. Dubbed *the prime frame of reference,* the two frames of reference are shown to the right.



NOTE: If you haven't yet been told or figured out why the frames are positioned as they are, don't worry. That explanation is coming. For now, just accept that that is what they look like when superimposing on one another like this.

There is a problem, though, with this grid. It is misleading. Lines of simultaneity are supposed to be *parallel* to the position axis. They are for the unprimed axis but not for the primed axis. If we wanted to take data from the primed axis, we have to realign the grid so that the required parallel lines are in evidence. That is done on the graph on the next page.



Before doing so, though, please remember that the interval is such that:

$$\tau^2 = \Delta x^2 - (c\Delta t)^2 \qquad \text{or} \qquad \tau^2 = (c\Delta t)^2 - \Delta x^2$$

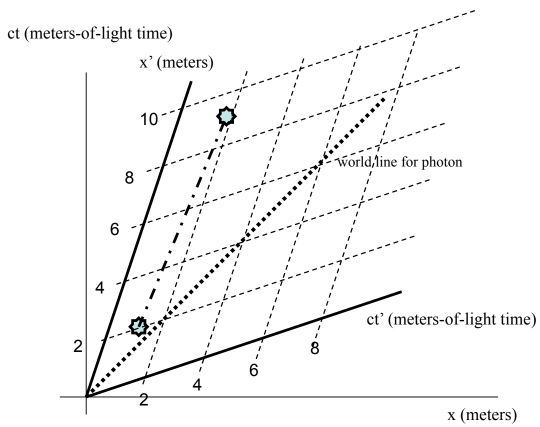
depending upon which right-hand term is larger.

With the grid squashed:

a.) We have lines of simultaneity being parallel with the x'-axis, as they should be.

b.) The *world line* of a photon of light is still at 45 degrees.

c.) The interval hasn't changed--the events are the same "distance" from one another as they always were. What more, we can calculate that interval using,

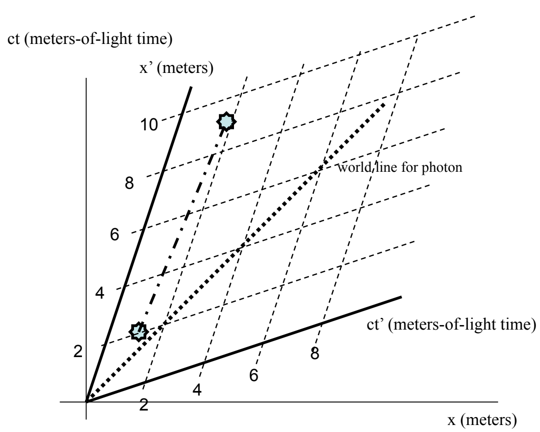


$$\tau^2 = (c\Delta t')^2 - (\Delta x')^2$$

just like before using the other coordinate axes.

Put in a little different context, it is true that:

$$\tau^2 = (c\Delta t')^2 - (\Delta x')^2$$
$$= (c\Delta t)^2 - (\Delta x)^2 .$$



Minor notes:

a.) If there is enough time for light to move from the first event to the second event (that is to say, if the slope of the line connecting the two events is greater than 1—where 1 is the slope of a photon's path through the region), the interval between the two events is referred to as a time-like interval and the interval is defined as:

$$c^2\Delta t^2 - \Delta x^2 = \tau^2$$

b.) If there is exactly enough time for light to move from the first event to the second event (that is to say, if the slope of the line connecting the two events is equal to 1), the interval between the two events is referred to as a light-like interval.

c.) If there is NOT enough time for light to move from the first event to the second event (that is to say, if the slope of the line connecting the two events is less than 1), the interval between the two events is referred to as a space-like interval and the interval is written as:

$$\Delta x^2 - c^2\Delta t^2 = \tau^2$$

Again, in all cases the interval is positive.